Small-Step Structural Operational Semantics (Small-Step SOS)

- Gordon Plotkin (1981), under the name *natural semantics*
- Also known as *transitional semantics*, or *reduction semantics*
- One can regard a small-step SOS as a device capable of executing a program step-by-step
- *Configuration*: tuple containing code and semantic ingredients
  - E.g., $\langle a, \sigma \rangle$ $\langle b, \sigma \rangle$ $\langle s, \sigma \rangle$ $\langle p \rangle$
- *Sequent (transition)*: Pair of configurations, to be *derived (proved)*
  - E.g., $\langle a_1, \sigma \rangle \to \langle a_1', \sigma \rangle$ $\langle a_1 + a_2, \sigma \rangle \to \langle a_1' + a_2, \sigma \rangle$
- *Rule*: Tells how to derive a sequent from others
  - E.g.,

$$\frac{\langle a_1, \sigma \rangle \to \langle a_1', \sigma \rangle}{\langle a_1 + a_2, \sigma \rangle \to \langle a_1' + a_2, \sigma \rangle}$$

Small-Step SOS of IMP - Arithmetic

State lookup

$$\langle x, \sigma \rangle \rightarrow \langle \boxed{\sigma(x)}, \sigma \rangle \qquad \text{(SmallStep-Lookup)}$$

$$\frac{\langle a_1, \sigma \rangle \rightarrow \langle a_1', \sigma \rangle}{\langle a_1 + a_2, \sigma \rangle \rightarrow \langle a_1' + a_2, \sigma \rangle} \qquad \text{(SmallStep-Add-Arg1)}$$

$$\frac{\langle a_2, \sigma \rangle \rightarrow \langle a_2', \sigma \rangle}{\langle a_1 + a_2, \sigma \rangle \rightarrow \langle a_1 + a_2', \sigma \rangle} \qquad \text{(SmallStep-Add-Arg2)}$$

$$\langle i_1 + i_2, \sigma \rangle \rightarrow \langle i_1 +_{Int} i_2, \sigma \rangle \qquad \text{(SmallStep-Add)}$$

Small-Step SOS of IMP - Arithmetic

$$\frac{\langle a_1, \sigma \rangle \to \langle a_1', \sigma \rangle}{\langle a_1 \,/\, a_2, \sigma \rangle \to \langle a_1' \,/\, a_2, \sigma \rangle} \qquad (\textsc{SmallStep-Div-Arg1})$$

$$\frac{\langle a_2, \sigma \rangle \to \langle a_2', \sigma \rangle}{\langle a_1 \,/\, a_2, \sigma \rangle \to \langle a_1 \,/\, a_2', \sigma \rangle} \qquad (\textsc{SmallStep-Div-Arg2})$$

$$\langle i_1 \,/\, i_2, \sigma \rangle \to \langle i_1 \,/_{\mathit{Int}}\, i_2, \sigma \rangle \quad \text{when } i_2 \neq 0 \qquad (\textsc{SmallStep-Div})$$

Small-Step SOS of IMP - Boolean

$$\frac{\langle a_1, \sigma \rangle \rightarrow \langle a_1', \sigma \rangle}{\langle a_1 \texttt{<=} a_2, \sigma \rangle \rightarrow \langle a_1' \texttt{<=} a_2, \sigma \rangle} \qquad (\textsc{SmallStep-Leq-Arg1})$$

$$\frac{\langle a_2, \sigma \rangle \rightarrow \langle a_2', \sigma \rangle}{\langle i_1 \texttt{<=} a_2, \sigma \rangle \rightarrow \langle i_1 \texttt{<=} a_2', \sigma \rangle} \qquad (\textsc{SmallStep-Leq-Arg2})$$

$$\langle i_1 \texttt{<=} i_2, \sigma \rangle \rightarrow \langle i_1 \leq_{Int} i_2, \sigma \rangle \qquad (\textsc{SmallStep-Leq})$$

Small-Step SOS of IMP - Boolean

$$\frac{\langle b, \sigma \rangle \to \langle b', \sigma \rangle}{\langle \texttt{not}\, b, \sigma \rangle \to \langle \texttt{not}\, b', \sigma \rangle} \qquad (\textsc{SmallStep-Not-Arg})$$

$$\langle \texttt{not true}, \sigma \rangle \to \langle \texttt{false}, \sigma \rangle \qquad (\textsc{SmallStep-Not-True})$$

$$\langle \texttt{not false}, \sigma \rangle \to \langle \texttt{true}, \sigma \rangle \qquad (\textsc{SmallStep-Not-False})$$

$$\frac{\langle b_1, \sigma \rangle \to \langle b_1', \sigma \rangle}{\langle b_1 \,\textsf{and}\, b_2, \sigma \rangle \to \langle b_1' \,\textsf{and}\, b_2, \sigma \rangle} \qquad (\textsc{SmallStep-And-Arg1})$$

$$\langle \texttt{false and}\, b_2, \sigma \rangle \to \langle \texttt{false}, \sigma \rangle \qquad (\textsc{SmallStep-And-False})$$

$$\langle \texttt{true and}\, b_2, \sigma \rangle \to \langle b_2, \sigma \rangle \qquad (\textsc{SmallStep-And-True})$$

Small-Step SOS Derivation

The following is a valid proof derivation, or proof tree, using the small-step SOS proof system for expressions of IMP above. Suppose that x and y are identifiers and $\sigma(x)=1$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\cdot}{\langle x, \sigma\rangle \rightarrow \langle 1, \sigma\rangle}}{\langle y \,/\, x, \sigma\rangle \rightarrow \langle y \,/\, 1, \sigma\rangle}}{\langle x + (y \,/\, x), \sigma\rangle \rightarrow \langle x + (y \,/\, 1), \sigma\rangle}}{\langle (x + (y \,/\, x)) \texttt{<=} x, \sigma\rangle \rightarrow \langle (x + (y \,/\, 1)) \texttt{<=} x, \sigma\rangle}$$

Small-Step SOS of IMP - Statements

$$\frac{\langle a, \sigma \rangle \rightarrow \langle a', \sigma \rangle}{\langle x := a, \sigma \rangle \rightarrow \langle x := a', \sigma \rangle} \qquad \text{(SmallStep-Asgn-Arg2)}$$

State update

$$\langle x := i, \sigma \rangle \rightarrow \langle \mathtt{skip}, \sigma[i/x] \rangle \qquad \text{(SmallStep-Asgn)}$$

$$\frac{\langle s_1, \sigma \rangle \rightarrow \langle s_1', \sigma' \rangle}{\langle s_1 \,;\, s_2, \sigma \rangle \rightarrow \langle s_1' \,;\, s_2, \sigma' \rangle} \qquad \text{(SmallStep-Seq-Arg1}$$

$$\langle \mathtt{skip} \,;\, s_2, \sigma \rangle \rightarrow \langle s_2, \sigma \rangle \qquad \text{(SmallStep-Seq-Skip)}$$

Small-Step SOS of IMP - Statements

$$\frac{\langle b, \sigma\rangle \to \langle b', \sigma\rangle}{\langle\, \texttt{if}\, b\, \texttt{then}\, s_1\, \texttt{else}\, s_2, \sigma\rangle \to \langle\, \texttt{if}\, b'\, \texttt{then}\, s_1\, \texttt{else}\, s_2, \sigma\rangle} \qquad (\textsc{SmallStep-If-Arg1})$$

$$\langle\, \texttt{if true then}\, s_1\, \texttt{else}\, s_2, \sigma\rangle \to \langle s_1, \sigma\rangle \qquad (\textsc{SmallStep-If-True})$$

$$\langle\, \texttt{if false then}\, s_1\, \texttt{else}\, s_2, \sigma\rangle \to \langle s_2, \sigma\rangle \qquad (\textsc{SmallStep-If-False})$$

$$\langle\, \texttt{while}\, b\, \texttt{do}\, s, \sigma\rangle \to \langle\, \texttt{if}\, b\, \texttt{then}\, (s\ ;\ \texttt{while}\, b\, \texttt{do}\, s)\, \texttt{else skip}\, , \sigma\rangle \qquad (\textsc{SmallStep-While})$$

$$\langle\, \texttt{vars}\, xl\ ;\ s\rangle \to \langle s, (xl \mapsto 0)\rangle \qquad (\textsc{SmallStep-Vars})$$

State initialization

Small-Step SOS in Rewriting Logic

- Any small-step SOS can be associated a rewrite logic theory (or, equivalently, a Maude module)
- The idea is to associate to each small-step SOS rule

$$\frac{C_1 \to C_1' \qquad C_2 \to C_2' \qquad \ldots \qquad C_n \to C_n'}{C \to C'}$$

a rewrite rule

$$\circ\overline{C} \to \overline{C'} \quad \textbf{if} \quad \circ\overline{C_1} \to \overline{C_1'} \ \wedge\ \circ\overline{C_2} \to \overline{C_2'} \ \wedge\ \ldots\ \wedge\ \circ\overline{C_n} \to \overline{C_n'}$$

(the circle means “ready for one step”)

Small-Step SOS of IMP in Maude

- See file **`imp-semantics-smallstep.maude`**